

An application of generalized Matlis duality for quasi- \mathcal{F} -modules to the Artinianness of local cohomology modules

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Abstract

We use a result of Hellus about generalized local duality to describe some generalized Matlis duals for certain quasi- \mathcal{F} -modules. Furthermore, we apply this description to obtain examples for non-artinian local cohomology modules by the theory of \mathcal{F} -modules. In particular, we get a new view on Hartshorne's counterexample for a conjecture by Grothendieck about the finiteness of $\operatorname{Hom}_R(R/I, H_I^i(R))$ for a noetherian local ring R and an ideal $I \subseteq R$.

1 Introduction

In 1992, Huneke [Hu92] stated four basic problems about local cohomology. One of these is the question whether or not a given local cohomology module is artinian.

Let $(R, \mathfrak{m}, \mathbb{k})$ be a noetherian local ring and M a finitely generated R -module. Then it is well-known that the local cohomology module $H_{\mathfrak{m}}^i(M)$ with support in the maximal ideal \mathfrak{m} is artinian for all i . On the other hand this is equivalent to both of the statements $\operatorname{Supp}_R(M) \subseteq \{\mathfrak{m}\}$ and the fact that $\operatorname{Hom}_R(R/\mathfrak{m}, M)$ is finitely generated. Regarding this, Grothendieck conjectured the following.

Conjecture 1.1 (Exposé XIII/Conjecture 1.1 in [Gro68]). *Let (R, \mathfrak{m}) be a noetherian local ring, $I \subseteq R$ an ideal and M a finitely generated R -module, then $\operatorname{Hom}_R(R/I, H_I^i(M))$ is finitely generated for all $n \in \mathbb{N}$.*

But in [Ha70] Hartshorne showed this to be false even for regular rings R by giving a counterexample. He showed that for the ring $R[[u, v, x, y]]$ and the ideals $\mathfrak{a} = (ux + vy)R$ and $I = (u, v)R$ the module $\operatorname{Hom}_R(R/\mathfrak{m}, H_I^2(R/\mathfrak{a}))$ is not finitely generated, and $\operatorname{Hom}_R(R/I, H_I^2(R/\mathfrak{a}))$ cannot be finitely generated either. In particular, $H_I^2(R/\mathfrak{a})$ is not artinian.

This example was generalized by Stückrad and Hellus in [HS09] to all modules of the form R/p with a prime $p \in (x, y)$. They used the fact that certain Matlis duals have infinitely many associated primes. In fact, they proved the following theorem.

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Theorem 1.2. *Let \mathbb{k} be any field, $R = \mathbb{k}[[X_1, \dots, X_n]]$ the ring of formal power series in the variables X_1, \dots, X_n ($n \geq 4$) and \mathfrak{a} the ideal (X_1, \dots, X_{n-2}) of R . If $p \in R$ is prime with $p \in (X_{n-1}, X_n)$ of R , then*

$$H_{\mathfrak{a}}^{n-2}(R/pR)$$

is not artinian.

Proof. [HS09, Theorem 2.4] □

In this paper we only consider the case of prime characteristic. But so we can find some new relations between the theory of \mathcal{F} -modules which was firstly introduced by Lyubeznik in [Lyu97] and Hartshorne's Example. We will show that we can translate the question whether certain local cohomology modules over a ring of formal power series are artinian into the task to decide whether a given \mathcal{F} -module is \mathcal{F} -finite.

More precisely we will prove:

Theorem (Theorem 4.3). *Let $(R, \mathfrak{m}, \mathbb{k})$ be a complete regular local ring of characteristic $p > 0$ with perfect residue field \mathbb{k} and let $I, \mathfrak{a} \subseteq R$ be ideals of R . If furthermore R/\mathfrak{a} is Cohen-Macaulay, we have for all $i \in \{0, \dots, \text{height } \mathfrak{a}\}$*

$$\mathfrak{D}(H_{\mathfrak{a}}^{\text{height } \mathfrak{a}-i}(R/I)) \cong H_I^i(D(H_{\mathfrak{a}}^{\text{height } \mathfrak{a}}(R))).$$

We will see that this R -module is an \mathcal{F} -module, but if it is not \mathcal{F} -finite the local cohomology module $H_{\mathfrak{a}}^{\text{height } \mathfrak{a}-i}(R/I)$ cannot be artinian.

The main ingredients for this theorem are a generalized local duality, formulated by Hellus in his habilitation thesis ([He07]) and an extension of usual Matlis duality to the category of quasi- \mathcal{F} -modules, which was firstly formulated by Lyubeznik in [Lyu97] for cofinite modules and later generalized by Blickle in [Bli01].

2 Generalized local duality

For a complete local ring (R, \mathfrak{m}) Matlis duality provides a correspondence between the category of noetherian R -modules and the category of artinian R -modules. So it is a quite interesting question which finitely generated modules correspond to the artinian local cohomology module $H_{\mathfrak{m}}^i(M)$ for a finitely generated R -module M . The local duality theorem answers this question.

Theorem 2.1 (local duality). *Let $(R, \mathfrak{m}, \mathbb{k})$ be a local d -dimensional Cohen-Macaulay ring with canonical module ω_R . Let M be a finitely generated R -module. Then we have for all $0 \leq i \leq d$:*

$$H_{\mathfrak{m}}^i(M) \cong D_R(\text{Ext}_R^{d-i}(M, \omega_R)).$$

Proof. [Iy07, Theorem 11.44]. □

In the above version of local duality, we have to consider for a local ring (R, \mathfrak{m}) the local cohomology modules with support in \mathfrak{m} . In [He07] Hellus could generalize this to a wider class of support-ideals under certain assumptions.

Theorem 2.2 (generalized local duality). *Let (R, \mathfrak{m}) be a noetherian local ring, $I \subseteq R$ an ideal and $h \in \mathbb{N}$, such that*

$$H_I^l(R) \neq 0 \iff l = h,$$

and let M be an R -module. then for all $i \in \{0, \dots, h\}$ we have a natural isomorphism

$$D(H_I^{h-i}(M)) \cong \text{Ext}_R^i(M, D(H_I^h(R))).$$

Proof. [He07, Theorem 6.4.1]. □

The next corollary shows that this is just a generalization of the usual local duality.

Corollary 2.3. *Let (R, \mathfrak{m}) be a noetherian complete local Cohen-Macaulay ring and let M be a finitely generated R -module. Additionally let $\omega_R := D(H_{\mathfrak{m}}^{\dim(R)}(R))$. Then we have an isomorphism*

$$H_{\mathfrak{m}}^{\dim(R)-i}(M) \cong D(\text{Ext}_R^i(M, \omega_R)).$$

Proof. [He07, Remark 6.4.2] □

3 Generalized Matlis-duality for quasi- \mathcal{F} -modules

Usual Matlis duality over a complete local ring (R, \mathfrak{m}) allows facts about artinian R -modules to be translated into corresponding statements about noetherian R -modules. Motivated by Lyubeznik's functor $\mathcal{H}_{R,A}$ for cofinite $R\{f\}$ -modules, presented in his influential paper [Lyu97], Blickle was able to extend the usual Matlis duality, defined as $D(M) := \text{Hom}_R(M, E(R/\mathfrak{m}))$, in [Bli01] to the category of quasi- \mathcal{F} -modules. He showed that this duality functor extends to a functor $D_{\mathcal{F}}: \text{quasi-}\mathcal{F}\text{-mod} \rightarrow \text{quasi-}\mathcal{F}\text{-mod}$, which involves Frobenius structures. We follow Blickle in [Bli01], who constructed this functor to analyze some duality properties for quasi- \mathcal{F} -modules. In the case of a complete regular local ring $(R, \mathfrak{m}, \mathbb{k})$ with perfect residue field we get in a functor

$$\mathfrak{D}: \mathcal{QF}\text{-mod} \longrightarrow \mathcal{F}\text{-mod}.$$

Let R be a noetherian commutative ring of positive characteristic p . The Frobenius homomorphism $\varphi: R \rightarrow R, r \mapsto r^p$ provides R with a nontrivial R -bimodule structure, given by the usual left action and the right action by the Frobenius. Therewith the Frobenius functor $\mathcal{F}: R\text{-mod} \rightarrow R\text{-mod}$ is defined due to Peskine and Szpiro in [PS73]:

$$\mathcal{F}(M) = R^{\varphi} \otimes_R M$$

$$\mathcal{F}\left(M \xrightarrow{f} N\right) = \left(R^{\varphi} \otimes_R M \xrightarrow{\text{id} \otimes_R f} R^{\varphi} \otimes_R N\right).$$

Now we are able to introduce the notation of a quasi- \mathcal{F} -module which is inspired by the $R[F]$ -modules in [Bli01], resp. by the $R\{f\}$ -modules in [Lyu97]. By avoiding these notions we particularly want to emphasise the relation to Lyubeznik's \mathcal{F} -modules. Nonetheless all these definitions are equivalent (see e.g. [Bli01, section 2.2]).

Definition 3.1 (quasi- \mathcal{F} -module). A **quasi- \mathcal{F} -module** is a pair (M, β) , consisting of an R -module M and a R -linear map

$$\beta: \mathcal{F}(M) = R^{\varphi} \otimes_R M \rightarrow M,$$

which we call **structure morphismus** of M . A morphism between two quasi- \mathcal{F} -modules (M, β) and (M', β') is an R -module homomorphism $f: M \rightarrow M'$, such that the following diagram commutes

$$\begin{array}{ccc} M & \xrightarrow{f} & M' \\ \downarrow \beta & & \downarrow \beta' \\ \mathcal{F}(M) & \xrightarrow{\mathcal{F}(f)} & M'. \end{array}$$

A quasi- \mathcal{F} -module (M, β) is called **\mathcal{F} -module** iff β is an isomorphism and we call an \mathcal{F} -module (M, β) **\mathcal{F} -finite** iff we could obtain the module M by a direct limit process of the form

$$M = \varinjlim \left(N \xrightarrow{\theta} \mathcal{F}(N) \xrightarrow{\mathcal{F}(\theta)} \mathcal{F}^2(N) \xrightarrow{\mathcal{F}^2(\theta)} \mathcal{F}^3(N) \xrightarrow{\mathcal{F}^3(\theta)} \dots \right)$$

with $N \in R\text{-mod}$ finitely generated.

Let $(R, \mathfrak{m}, \mathbb{k})$ be a complete regular local ring of characteristic $p > 0$ and let (M, β) be a quasi- \mathcal{F} -module. In this situation, by Cohen's structure theorem ([Iy07, Theorem 8.28]), R is isomorphic to a ring of formal power series in finitely many variables over the field \mathbb{k} . If, on top of this, \mathbb{k} is perfect, i.e. $\mathbb{k}^p = \mathbb{k}$, R is finitely generated over R^p . This means that R is a so-called F -finite ring and, by [Bli01, corollary 4.10], we get a natural isomorphism $\mathcal{F}(\text{Hom}_R(M, N)) \cong \text{Hom}_R(\mathcal{F}(M), \mathcal{F}(N))$ for all R -modules M and N . Since the injective hull of the residue field $E(R/\mathfrak{m})$ is in fact an \mathcal{F} -module, i.e. $\mathcal{F}(E(R/\mathfrak{m})) \cong E(R/\mathfrak{m})$, we get an isomorphism

$$\tau_M : \mathcal{F}(\text{Hom}_R(M, E(R/\mathfrak{m}))) = \mathcal{F}(D(M)) \cong D(\mathcal{F}(M)) = \text{Hom}_R(\mathcal{F}(M), E(R/\mathfrak{m})),$$

for all R -modules M . Matlis duality yields a map

$$\gamma : D(M) \xrightarrow{D(\beta)} D(\mathcal{F}(M)) \xrightarrow{\tau_M} \mathcal{F}(D(M)).$$

With this map, Blickle defined the following functor (see [Bli01, section 4.2] for details).

Definition 3.2. Let $(R, \mathfrak{m}, \mathbb{k})$ be a complete regular local ring of positive characteristic $p > 0$ and let (M, β) be a quasi- \mathcal{F} -module (finitely generated or artinian as R -module if \mathbb{k} is not perfect). Let $\gamma := \tau_M \circ D(\beta)$. Then

$$\mathfrak{D}(M) := \varinjlim \left(D(M) \xrightarrow{\gamma} \mathcal{F}(D(M)) \xrightarrow{\mathcal{F}(\gamma)} \mathcal{F}^2(D(M)) \rightarrow \dots \right)$$

is an \mathcal{F} -module generated by γ . On the above-mentioned class of modules (resp. rings) this construction defines an exact functor.

The exactness is obvious by the exactness of the usual Matlis duality functor and the direct limit. If, even more, M is an \mathcal{F} -module, hence γ an isomorphism, the direct system only consists of one element and we have $\mathfrak{D}(M) = D(M)$. If M is artinian, by Matlis duality $D(M)$ is finitely generated and $\mathfrak{D}(M)$ is in fact \mathcal{F} -finite.

Remark 3.3. If the quasi- \mathcal{F} -module (M, β) is an \mathcal{F} -module, i.e. $\mathcal{F}(M) \cong M$, over the complete regular local ring $(R, \mathfrak{m}, \mathbb{k})$ with perfect residue field \mathbb{k} , we see that the map τ_M from above yields an \mathcal{F} -module structure on the Matlis dual $D(M) := \text{Hom}_R(M, E(R/\mathfrak{m}))$. In fact by precomposing with β^{-1} we get an isomorphism

$$\mathcal{F}(D(M)) \cong D(\mathcal{F}(M)) \cong D(M).$$

From [Lyu97, 2.12] we know that \mathcal{F} -finite modules only have finitely many associated primes. In contrast to this, in [BN08, 3.5] it is shown that the Matlis duals $D(H_{(X_1, \dots, X_i)}^i(\mathbb{k}[[X_1, \dots, X_n]]))$ have infinitely many associated primes for $i \leq n$. So Matlis duals of \mathcal{F} -finite R -modules over complete regular local rings with perfect residue field are also \mathcal{F} -modules which generally are not \mathcal{F} -finite.

Now, as a first example, we are going to describe the generalized Matlis dual of the top local cohomology module $H_{\mathfrak{m}}^d(R/I)$ of a quotient of a complete regular local ring R . We should keep in mind that the module $H_{\mathfrak{m}}^d(R/I)$ is just a quasi- \mathcal{F} -module, but the generalized Matlis dual $\mathfrak{D}(H_{\mathfrak{m}}^d(R/I))$ will provide us with an \mathcal{F} -finite module.

Example 3.4. [Bli01, 4.3.2] Let $(R, \mathfrak{m}, \mathbb{k})$ be a complete regular local ring of characteristic $p > 0$ and dimension n . Let furthermore $I \subseteq R$ be an ideal of R with height $I = n - d = c$ and let $S := R/I$. Then S is a ring of dimension d and the local cohomology module $H_{\mathfrak{m}}^i(S)$ is an \mathcal{F} -module when considered as module over S . If we consider $H_{\mathfrak{m}}^i(S)$ as an R -module it is not generally not an \mathcal{F} -module, but only a quasi- \mathcal{F} -module with structure morphism

$$\beta : R^\varphi \otimes_R H_{\mathfrak{m}}^i(R/I) \rightarrow H_{\mathfrak{m}}^i(R/I).$$

This map is equivalent to the map induced by the projection $R/I^{[p]} \rightarrow R/I$ under the identification of $R^\varphi \otimes_R H_{\mathfrak{m}}^i(R/I)$ with $H_{\mathfrak{m}}^i(R/I^{[p]})$. By definition $\mathfrak{D}(H_{\mathfrak{m}}^i(R/I))$ is the limit of the direct system

$$D(H_{\mathfrak{m}}^i(R/I)) \rightarrow D(H_{\mathfrak{m}}^i(R/I^{[p]})) \rightarrow D(H_{\mathfrak{m}}^i(R/I^{[p^2]})) \rightarrow \dots$$

Now we can use the local duality for complete local Gorenstein rings (see [Iy07, 11.29]) since R is regular and local. We get an isomorphism of direct systems

$$\begin{array}{ccccccc} D(H_{\mathfrak{m}}^i(R/I)) & \longrightarrow & D(H_{\mathfrak{m}}^i(R/I^{[p]})) & \longrightarrow & D(H_{\mathfrak{m}}^i(R/I^{[p^2]})) & \longrightarrow & \dots \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\ \text{Ext}_R^{n-i}(R/I, R) & \longrightarrow & \text{Ext}_R^{n-i}(R/I^{[p]}, R) & \longrightarrow & \text{Ext}_R^{n-i}(R/I^{[p^2]}, R) & \longrightarrow & \dots \end{array}$$

The maps in the bottom system are induced by the natural projections and thus we have (see i.e. [Iy07, Remark 7.9] for the last isomorphism)

$$\mathfrak{D}(H_{\mathfrak{m}}^i(R/I)) = \varinjlim_k D(H_{\mathfrak{m}}^i(R/I^{[p^k]})) \cong \varinjlim_k \text{Ext}_R^{n-i}(R/I^{[p^k]}, R) \cong \varinjlim_k \text{Ext}_R^{n-i}(R/I^k, R).$$

So, all in all we can formulate the following:

Theorem 3.5. Let $(R, \mathfrak{m}, \mathbb{k})$ be a complete regular local ring of characteristic $p > 0$, \mathbb{k} perfect and $\dim R = n$. Let also $I \subseteq R$ be an ideal of R of height $c = n - d$. Then one has

$$\mathfrak{D}(H_{\mathfrak{m}}^i(R/I)) \cong H_I^{n-i}(R)$$

as \mathcal{F} -modules. In particular for the top local cohomology module we have an isomorphism

$$\mathfrak{D}(H_{\mathfrak{m}}^d(R/I)) \cong H_I^c(R).$$

Proof. By the characterization of local cohomology as a direct limit of certain Ext -modules we get

$$\varinjlim_k \text{Ext}_R^{n-i}(R/I^k, R) \cong H_I^{n-i}(R).$$

□

4 Result

Our aim is now to use the generalized local duality from section 2 to obtain a description of generalized Matlis duals \mathfrak{D} for certain local cohomology quasi- \mathcal{F} -modules, which are more general than those from the above example. In the example we used usual local duality to describe modules of the form $\mathfrak{D}(H_{\mathfrak{m}}^i(R/I))$. By the results of Hellus we can now, under special assumptions, also describe modules of the form $\mathfrak{D}(H_{\mathfrak{a}}^i(R/I))$.

Theorem 4.1. *Let $(R, \mathfrak{m}, \mathbb{k})$ be a complete regular local ring of characteristic $p > 0$ with \mathbb{k} perfect and let $I, \mathfrak{a} \subseteq R$ be ideals of R . Furthermore, let $h \in \mathbb{N}$ be chosen in a way that $H_{\mathfrak{a}}^l(R) \neq 0 \Leftrightarrow l = h$. Then for all $i \in \{0, \dots, h\}$ there is an isomorphism*

$$\mathfrak{D}(H_{\mathfrak{a}}^{h-i}(R/I)) \cong H_I^i(D(H_{\mathfrak{a}}^h(R))).$$

Proof. With the given assumptions, we have by Theorem 2.2

$$D(H_{\mathfrak{a}}^{h-i}(R/I^{[p^k]})) \cong \text{Ext}_R^i(R/I^{[p^k]}, D(H_{\mathfrak{a}}^h(R))).$$

Now, by Definition 3.2 follows

$$\begin{aligned} \mathfrak{D}(H_{\mathfrak{a}}^{h-i}(R/I)) &\cong \varinjlim_k D(H_{\mathfrak{a}}^{h-i}(R/I^{[p^k]})) \\ &\cong \varinjlim_k \text{Ext}_R^i(R/I^{[p^k]}, D(H_{\mathfrak{a}}^h(R))) \\ &\cong H_I^i(D(H_{\mathfrak{a}}^h(R))). \end{aligned}$$

□

Due to Peskine and Szpiro in [PS73], it is possible to formulate the following.

Lemma 4.2. *Let R be a regular domain of characteristic $p > 0$ and let $\mathfrak{a} \subseteq R$ be an ideal of R , such that R/\mathfrak{a} is Cohen-Macaulay. Then we obtain*

$$H_{\mathfrak{a}}^i(R) = 0 \quad \text{für } i \neq \text{height } \mathfrak{a}.$$

Proof. [Iy07, Theorem 21.29].

□

If we apply this lemma to our result (Theorem 4.1) we can give the following description of the generalized Matlis duals of certain local cohomology modules.

Theorem 4.3. *Let $(R, \mathfrak{m}, \mathbb{k})$ be a complete regular local ring of characteristic $p > 0$ with perfect residue field \mathbb{k} and let $I, \mathfrak{a} \subseteq R$ be ideals of R . If furthermore R/\mathfrak{a} is Cohen-Macaulay and $i \in \{0, \dots, \text{height } \mathfrak{a}\}$ is arbitrary, we have*

$$\mathfrak{D}(H_{\mathfrak{a}}^{\text{height } \mathfrak{a}-i}(R/I)) \cong H_I^i(D(H_{\mathfrak{a}}^{\text{height } \mathfrak{a}}(R))).$$

Proof. As a regular local ring R is Cohen-Macaulay and a domain (see [Iy07, 11.3; 11.10] and [Iy07, 8.18]). Hence, we have

$$\text{height } \mathfrak{a} = \text{depth}_R(\mathfrak{a}, R).$$

Thus, $H_{\mathfrak{a}}^{\text{height } \mathfrak{a}}(R) \neq 0$ and overall by lemma 4.2 we get

$$H_{\mathfrak{a}}^i(R) \neq 0 \iff i = \text{height } \mathfrak{a}.$$

The claim now follows from Theorem 4.1.

□

Remark 4.4. *If we set \mathfrak{a} to be the maximal ideal \mathfrak{m} of R and $\dim R = n$, we have $\text{height } \mathfrak{m} = n$ and R/\mathfrak{m} is Cohen-Macaulay. So Theorem 4.3 yields*

$$\mathfrak{D}(H_{\mathfrak{m}}^i(R/I)) \cong H_I^{n-i}(D(H_{\mathfrak{m}}^n(R))).$$

But since $H_{\mathfrak{m}}^n(R)$ is artinian, by local duality ([Iy07, 11.29]) and Matlis duality ([Iy07, A.35]) we know there are isomorphisms $D(H_{\mathfrak{m}}^n(R)) \cong \text{Ext}_R^0(R, R) = R$. And hence, Theorem 4.3 in fact generalizes Theorem 3.5.

5 An example

Now we are going to connect our new description of certain generalized Matlis duals to Hartshorne's example, resp. to the generalization of this example by Stückrad and Hellus (see Theorem 1.2).

By definition of generalized Matlis duality (3.2) we have the logical validity of the implication

$$H_I^i(M) \text{ artinian quasi-}\mathcal{F}\text{-module} \implies \mathfrak{D}(H_I^i(M)) \text{ } \mathcal{F}\text{-finite,}$$

and hence

$$\mathfrak{D}(H_I^i(M)) \text{ not } \mathcal{F}\text{-finite} \implies H_I^i(M) \text{ not artinian.}$$

Therefore we can check the \mathcal{F} -finiteness of $\mathfrak{D}(H_I^i(M))$ to get results about the Artinianness of $H_I^i(M)$. For the generalization of Hartshorne's example by Stückrad and Hellus (Theorem 1.2) we get in particular:

Example 5.1. Let $\mathbb{k} = \mathbb{F}_p$, $R = \mathbb{k}[[X_1, \dots, X_n]]$ the ring of formal powers series in X_1, \dots, X_n ($n \geq 4$) over the finite field \mathbb{k} and let \mathfrak{a} be the ideal (X_1, \dots, X_{n-2}) R . So we have

$$\text{height } \mathfrak{a} = n - 2$$

and

$$R/\mathfrak{a} \cong R[[X_{n-1}, X_n]]$$

is Cohen-Macaulay. So, by Theorem 4.3, we get for $i \in \{0, \dots, \text{height } \mathfrak{a}\}$ and an ideal $I \subseteq R$

$$\mathfrak{D}(H_{\mathfrak{a}}^{n-2-i}(R/I)) \cong H_I^i(D(H_{\mathfrak{a}}^{n-2}(R))).$$

In particular, if we take $i = 0$ and $I = (p)$ for some prime $p \in (X_{n-1}, X_n)$, we get

$$\mathfrak{D}(H_{\mathfrak{a}}^{n-2}(R/pR)) \cong H_{pR}^0(D(H_{\mathfrak{a}}^{n-2}(R))).$$

If we now consider the set of associated primes of the latter module, we see that

$$\text{Ass}(H_{pR}^0(D(H_{\mathfrak{a}}^{n-2}(R)))) = \text{Ass}(\Gamma_{pR}(D(H_{\mathfrak{a}}^{n-2}(R)))) = \{\mathfrak{q} \in \text{Ass}(D(H_{\mathfrak{a}}^{n-2}(R))) \mid p \in \mathfrak{q}\}.$$

But from [He07, 4.3.4], we know that at least in the case where $p \in \mathfrak{a}$, this set has infinitely many elements and so by [Lyu97, 2.12] the module

$$\mathfrak{D}(H_{\mathfrak{a}}^{n-2}(R/pR)) \cong H_{pR}^0(D(H_{\mathfrak{a}}^{n-2}(R)))$$

cannot be \mathcal{F} -finite. We have herewith reproven $H_{\mathfrak{a}}^{n-2}(R/pR)$ not to be artinian.

6 Further applications

As a first consequence of Theorem 4.3 we can give at least in the regular case a very short proof for [He07, Theorem 7.4.1].

Corollary 6.1. Let $(R, \mathfrak{m}, \mathbb{k})$ be a complete regular local ring of characteristic $p > 0$ with perfect residue field \mathbb{k} and $x_1, \dots, x_i \in R$ ($i \geq 1$) a regular sequence in R . Set $I := (x_1, \dots, x_i)R$. Then we have a natural isomorphism

$$H_I^i(D(H_I^i(R))) \cong E_R(\mathbb{k}).$$

Proof. I is a set-theoretic complete intersection ideal of R since it is generated by a regular sequence and so R/I is a complete intersection ring. Therefore, R/I is Cohen-Macaulay ([Iy07, 10.5]) and by Theorem 4.3 we have

$$H_I^i(D(H_I^i(R))) \cong \mathfrak{D}(H_I^0(R/I)).$$

Further, by the definition of the generalized Matlis duality, we conclude

$$\begin{aligned} \mathfrak{D}(H_I^0(R/I)) &= \varinjlim D(H_I^0(R/I)) \\ &= \varinjlim \operatorname{Hom}_R(H_I^0(R/I^{[p^k]}), E(\mathbb{k})) \\ &= \varinjlim \operatorname{Ext}_R^0(\Gamma_I(R/I^{[p^k]}), E(\mathbb{k})) \\ &= \varinjlim \operatorname{Ext}_R^0(R/I^{[p^k]}, E(\mathbb{k})) \\ &\cong H_I^0(E(\mathbb{k})) \\ &\cong \Gamma_I(E(\mathbb{k})) \\ &\cong E(\mathbb{k}). \end{aligned}$$

□

Example 6.2. Let $\mathbb{k} = \mathbb{F}_p$, $R = \mathbb{k}[[X_1, \dots, X_n]]$ the ring of formal powers series in X_1, \dots, X_n over the finite field \mathbb{k} and let I be the ideal (X_1, \dots, X_i) of R . Then, we have an isomorphism

$$H_I^i(D(H_I^i(R))) \cong E_R(\mathbb{k}).$$

As an application of the last corollary and the fact that certain Matlis duals of local cohomology modules are \mathcal{F} -modules (see Remark 3.3), we can extend [He07, Theorem 7.4.2] to the case of a local ring of prime characteristic with perfect coefficient field.

Theorem 6.3. Let $(R, \mathfrak{m}, \mathbb{k})$ be a complete regular local equicharacteristic ring with perfect residue field \mathbb{k} , $I \subseteq R$ an ideal of height $h \geq 1$, and assume that

$$H_I^l(R) = 0 \quad \forall l > h.$$

Then one has

$$H_I^h(D(H_I^h(R))) = E_R(\mathbb{k}) \quad \text{or} \quad H_I^h(D(H_I^h(R))) = 0.$$

Proof. If the characteristic of R is zero this is just the statement of [He07, Theorem 7.4.2]. So let R be a complete regular local ring of prime characteristic $p > 0$ with perfect residue field \mathbb{k} . We can use more or less the same arguments like in the characteristic zero case, but we have to consider the \mathcal{F} -module structure of the Matlis duals (see Remark 3.3) instead of the structure as a \mathcal{D} -module. Here are the details.

As a regular local ring R is Cohen-Macaulay and hence, we have $\operatorname{height} I = \operatorname{depth}_R(I, R)$. So let $x_1, \dots, x_h \in I$ be an R -regular sequence. If we set

$$D := D(H_{(x_1, \dots, x_h)R}^h(R)),$$

from [He07, 1.1.4], we know that x_1, \dots, x_h is also a D -regular sequence, so we have $\operatorname{depth}((x_1, \dots, x_h)R, D) \geq h$ and hence

$$H_{(x_1, \dots, x_h)R}^i(D) = 0 \quad \forall i < h. \tag{1}$$

If we now consider the Grothendieck spectral sequence for composed functors (see e.g. [Rot79, 10.47])

$$E_2^{p,q} = (R^p F) (R^q G) A \xrightarrow{p} R^n (FG) A,$$

we can apply this to the composition $\Gamma_I \circ \Gamma_{(x_1, \dots, x_h)R}$ and the module D and get

$$E_2^{p,q} = (H_I^p) \left(H_{(x_1, \dots, x_h)R}^q \right) (D) \xrightarrow{p} H^n \left(\Gamma_I \circ \Gamma_{(x_1, \dots, x_h)R} \right) (D)$$

By (1) this sequence collapses on the q -axis, since the only nonzero modules are $E_2^{p,h}$ and hence in consequence of [Rot79, 10.26]

$$(H_I^0) \left(H_{(x_1, \dots, x_h)R}^h \right) (D) \cong H^h \left(\Gamma_I \circ \Gamma_{(x_1, \dots, x_h)R} \right) (D).$$

As $H_I^0 \cong \Gamma_I$ and $(x_1, \dots, x_h)R \subseteq I$,

$$H_I^h(D) \cong \Gamma_I \left(H_{(x_1, \dots, x_h)R}^h(D) \right) \subseteq H_{(x_1, \dots, x_h)R}^h(D) \cong E_R(\mathbb{k}) \quad (2)$$

holds, where the last isomorphism is Corollary 6.1.

Since $E_R(\mathbb{k})$ is artinian (see [Iy07, A.32]) it is $\text{Supp } E(\mathbb{k}) = \{\mathfrak{m}\}$ and hence $\dim E(\mathbb{k}) = 0$. By (2) $H_I^h(D)$ is a submodule of $E_R(\mathbb{k})$ and so we have also $\dim H_I^h(D) = 0$. Because R is an \mathcal{F} -module by Remark 3.3 and [Lyu97, 1.2(b)] $H_I^h(D)$ is an \mathcal{F} -module and so [Lyu97, 1.4] yields

$$\text{inj dim } H_I^h(D) = 0.$$

So $H_I^h(D)$ is injective and therefore it is isomorphic to a direct sum of modules of the form $E_R(R/\mathfrak{p})$ for $\mathfrak{p} \in \text{Spec } R$. But as a submodule of $E_R(\mathbb{k})$ it has to be either $E_R(\mathbb{k})$ itself or zero and in addition, the natural injection

$$H_I^h(R) \subseteq H_{(x_1, \dots, x_h)R}^h(R)$$

induces a surjection

$$D \twoheadrightarrow D(H_I^h(R))$$

and due to the fact that $H_I^h(R)$ is right-exact, a surjection

$$H_I^h(D) \twoheadrightarrow H_I^h(D(H_I^h(R))).$$

But again the module on the right side is by Remark 3.3 and [Lyu97, 1.2(b);1.4] an injective \mathcal{F} -module and we have a surjection from zero or $E(\mathbb{k})$ to it. Therefore the direct sum decomposition consists at most of one copy of $E(\mathbb{k})$ and we get the asserted statement. □

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